New Information from New Maps

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Chicago
5 December 2007
Maps are Useful!
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Constellations
Maps are Useful!
Science or Stamp Collecting?

Context: Proposal for SDSS ⇒ SDSS-II

- Dominant view on how to obtain funding:
  » Parametric Cosmology
    - dark energy, m_\nu, ...

- Realistic view of how science works:
  » Mapping the Sky / Astronomical Infrastructure
  » Quality maps will produce large scientific yield.
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Make it and they will come!
Quantify New Information Obtained by Better Mapping
Cartography and Parameter Estimation are Mathematically Similar Measurements with Statistical Uncertainties!

- **Parametric Cosmology:**
  - $H_0$, $T_γ_0$, $Ω_{m0}$, $Ω_{b0}$, $m_ν$, $σ_8$, $n_s$, $n_t$, $r$, $w$, $w_A$...
  - $\sim 10^1$ pixels

- **Power Spectrum Estimation:**
  - e.g. $C^{TT}_\ell$
  - $\sim 10^3$ pixels

- **Cartography:**
  - $\sim 10^6$ pixels
Important Tool: Fisher Metric
a.k.a. Fisher Information Matrix

\[ F[p] = \langle (\nabla_x \ln[\mathcal{L}[x | d]]) (\nabla_x \ln[\mathcal{L}[x | d]]) \rangle_d \]
\[ = -\langle \nabla_x \nabla_x \ln[\mathcal{L}[x | d]] \rangle_d \]

- Inverse Fisher Matrix gives lower bound on errors of estimators

\( d \) - data
\( x \) - parameters or "map"
\( \mathcal{L} \) - likelihood function
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- Inverse Fisher Matrix gives lower bound on errors of estimators
- Fisher Matrix transforms like a tensor on parameter space manifold!
  - It is parameterization independent
- It is a positive-definite rank-2 tensor.
- Can be used as a metric on parameter space
  - e.g. space of possible maps

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Fisher Geometry!
What is Fisher Geometry?

- Fisher Distance:

\[ d_F[x_1, x_2] = \int_{\text{geodesic}} d\ell \sqrt{x'[\ell] \cdot F[x[\ell]] \cdot x'[\ell]} \]

- For your measurement (technique) \( d_F[x_1, x_2] \) gives the "# of
What is Fisher Geometry?

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- **Fisher Geometry is “Physics Free”**
  - depends on limitation of measurement
    - Noise (instrumental, sky, … )
    - Sky coverage
    - Overall limitations of technique.
Fisher Volume

- Fisher Volume for measurement (B)

\[ V_F^{(B)} = \int d^n x \sqrt{\text{Det}[F_B[x]]} \frac{\mathcal{L}_B[x]}{\sup[\mathcal{L}_B]} \]

» uninteresting
Fisher Volume

- Fisher Volume for measurement \((B)\)

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V_F^{(B)} = \int d^n x \sqrt{\text{Det}[F_B[x]]} \frac{L_B[x]}{\text{sup}[L_B]}
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» uninteresting

- Conditional Fisher Volume (measurement \(A\) given \(B\))

\[
V_F^{(A|B)} = \int d^n x \sqrt{\text{Det}[F_{A+B}[x]]} \frac{L_B[x]}{\text{sup}[L_B[x]]}
\]
Fisher Volume

- Fisher Volume for measurement (B)
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  \[ V_F^{(A|B)} = \int d^n x \sqrt{\text{Det}[F_{A+B}[x]]} \frac{L_B[x]}{\sup[L_B[x]]} \]

Roughly: # of distinguishable results obtainable from A+B which were consistent with measurement B.
Fisher Information Number

clean it up

Natural log of # of distinguishable results obtainable from A+B which are consistent with measurement B.

\[ I_F^{(A|B)} = \ln \left( \frac{V_F^{(A|B)}}{V_F^{(B)}} \right) \]
Fisher Information Number

Examples

- **Unconstrained map w/ Gaussian measurement “noise”**.
  - 2 measurements of same map with noise matrices $N_A$ and $N_B$.

\[
I^{(A|B)}_{\text{unconstrained}} = \frac{1}{2} \ln[\text{Det}[I + N_B \cdot N_A^{-1}]]
\]
Fisher Information Number

Examples

- Unconstrained map w/ Gaussian measurement “noise”.
  - 2 measurements of same map with noise matrices $N_A$ and $N_B$.

  $$I^{(A|B)}_{\text{unconstrained}} = \frac{1}{2} \ln[\det[I + N_B \cdot N_A^{-1}]]$$

- Gaussian map w/ Gaussian “noise” w/ known Signal
  - Signal matrix $S$ (e.g. $P[k]$ or $C_l$) with noise matrices $N_A, N_B$.

  $$I^{(A|B)}_{\text{Gaussian}} = \frac{1}{2} \ln\left[\frac{\det[I + S \cdot N_A^{-1} + S \cdot N_B^{-1}]}{\det[I + S \cdot N_B^{-1}]}\right]$$
Fisher Information Number

Examples

- **Unconstrained map w/ Gaussian measurement “noise”.**
  
  » 2 measurements of same map with noise matrices $N_A$ and $N_B$.

  \[
  I_{\text{unconstrained}}^{(A|B)} = \frac{1}{2} \ln \left[ \text{Det} \left[ I + N_B \cdot N_A^{-1} \right] \right]
  \]

- **Gaussian map w/ Gaussian “noise” w/ known Signal**

  » Signal matrix $S$ (e.g. $P[k]$ or $C_l$) with noise matrices $N_A$, $N_B$.

  \[
  I_{\text{Gaussian}}^{(A|B)} = \frac{1}{2} \ln \left[ \frac{\text{Det} \left[ I + S \cdot N_A^{-1} + S \cdot N_B^{-1} \right]}{\text{Det} \left[ I + S \cdot N_B^{-1} \right]} \right]
  \]

- **N.B.**

  \[
  \lim_{S \to \infty} I_{\text{Gaussian}}^{(A|B)} = I_{\text{unconstrained}}^{(A|B)}
  \]
Fisher Information Number

Warning

- **Fisher geometry is Physics Free (unbiased)**
  - Adding a new significant digit to the value of a well measured pixel is contributes just as much as measuring the 1st significant digit of a “new” pixel
    - A scientist might find one more useful than the other
  - It counts all pixels and angular scales equally.
    - A scientist may find some things more interesting than others.

- **Fisher information is geometric (unbiased)**
  - It doesn’t matter how you parameterize your map you will always get the same number.
    - If 2 quantities have a 1-1 relation, (e.g. density - potential) you get the same number.
      - This works even if the relation is non-linear!
Fisher Information Number

Examples

- Gaussian all sky maps

\[
I^{(A|B)}_{\text{Gaussian}} = \frac{1}{2} \sum_l (2l + 1) \ln \left[ \frac{1 + \frac{C^S_l}{C^N_l,A} + \frac{C^S_l}{C^N_l,B}}{1 + \frac{C^S_l}{C^N_l,B}} \right]
\]

\[
I^{(A|B)} \sim \left( \frac{\sigma^B_{\text{beam}}}{\sigma^A_{\text{beam}}} \right)^2 - 1 \sim 10^{5-6}
\]

\[
e^{I^{(A|B)}} \sim 10^{10^5}
\]
Fisher Information Number

Examples

- Gaussian all sky maps

\[
I_{\text{Gaussian}}^{(A|B)} = \frac{1}{2} \sum_l (2l + 1) \ln \left[ \frac{1 + \frac{C_l^S}{C_{N,A}^l} + \frac{C_l^S}{C_{N,B}^l}}{1 + \frac{C_l^S}{C_{N,B}^l}} \right]
\]

- For hi-S/N, beam limited (not noise limited) surveys

\[
I^{(A|B)} \sim \frac{\left(\frac{\sigma_B^{\text{beam}}}{\sigma_A}\right)^2}{\sigma_{\text{beam}}^2} - 1 \sim 10^{5-6}
\]

\[
e^{I^{(A|B)}} \sim 10^{10^5}
\]

» Increased number of resolved hi-S/N pixels dominates
Fisher Information Number

Summary

- Using the Fisher information metric a “geometrical” measure of the amount of information obtained by making new measurements.
  - The information measure independent of how one parameterized what one is measuring.

- This information measure is “physics free”.
  - This can be good - no scientist bias.
  - This can be bad - justify stamp-collecting.

- One can compute it for almost any type of measurement which one has a statistical model for
  - It may be difficult to compute!
  - It is easy for Gaussian systems (much of cosmology).