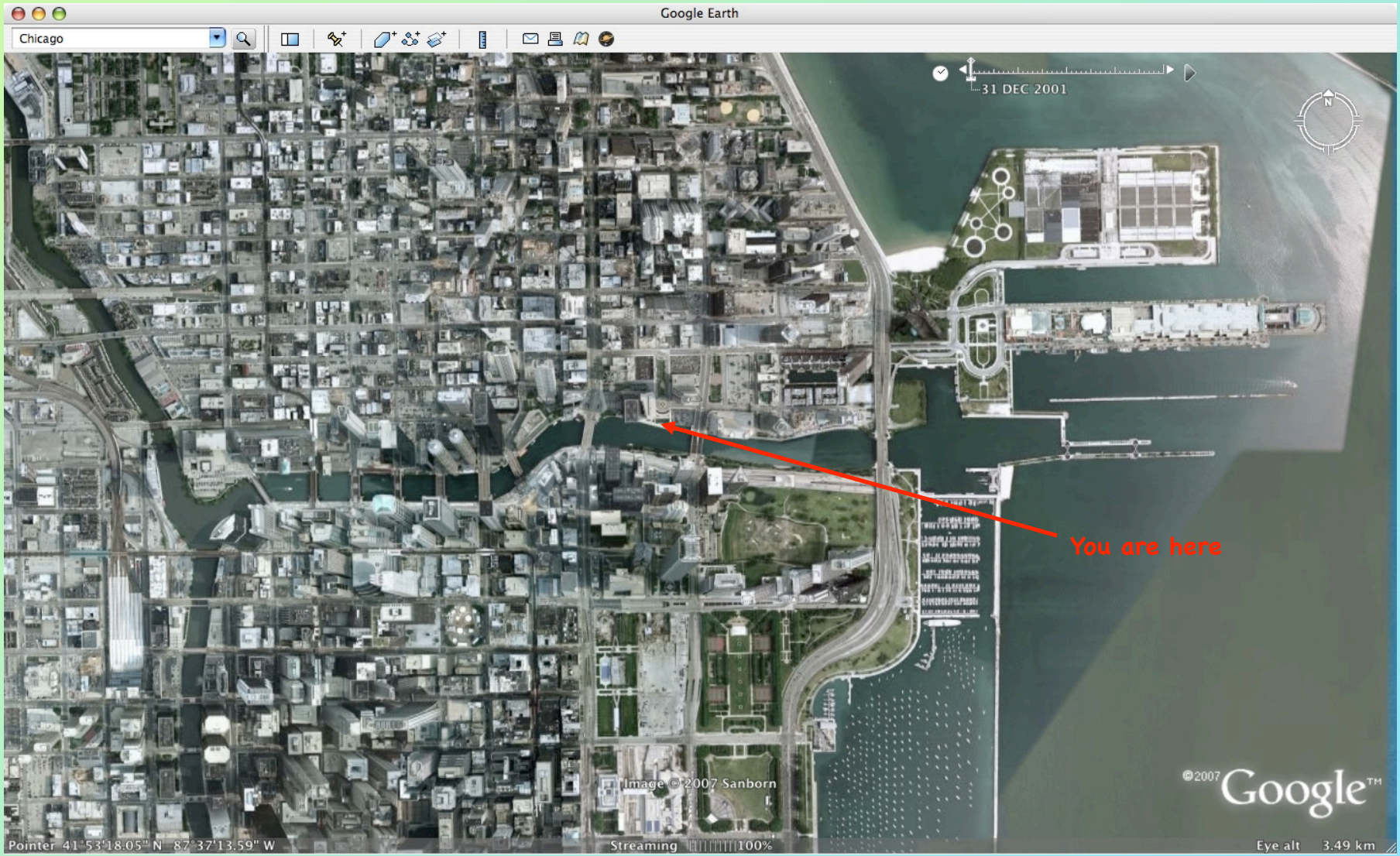
New Information from New Maps

Albert Stebbins
Fermilab

CosmiCartography
Chicago
5 December 2007

Maps are Useful!



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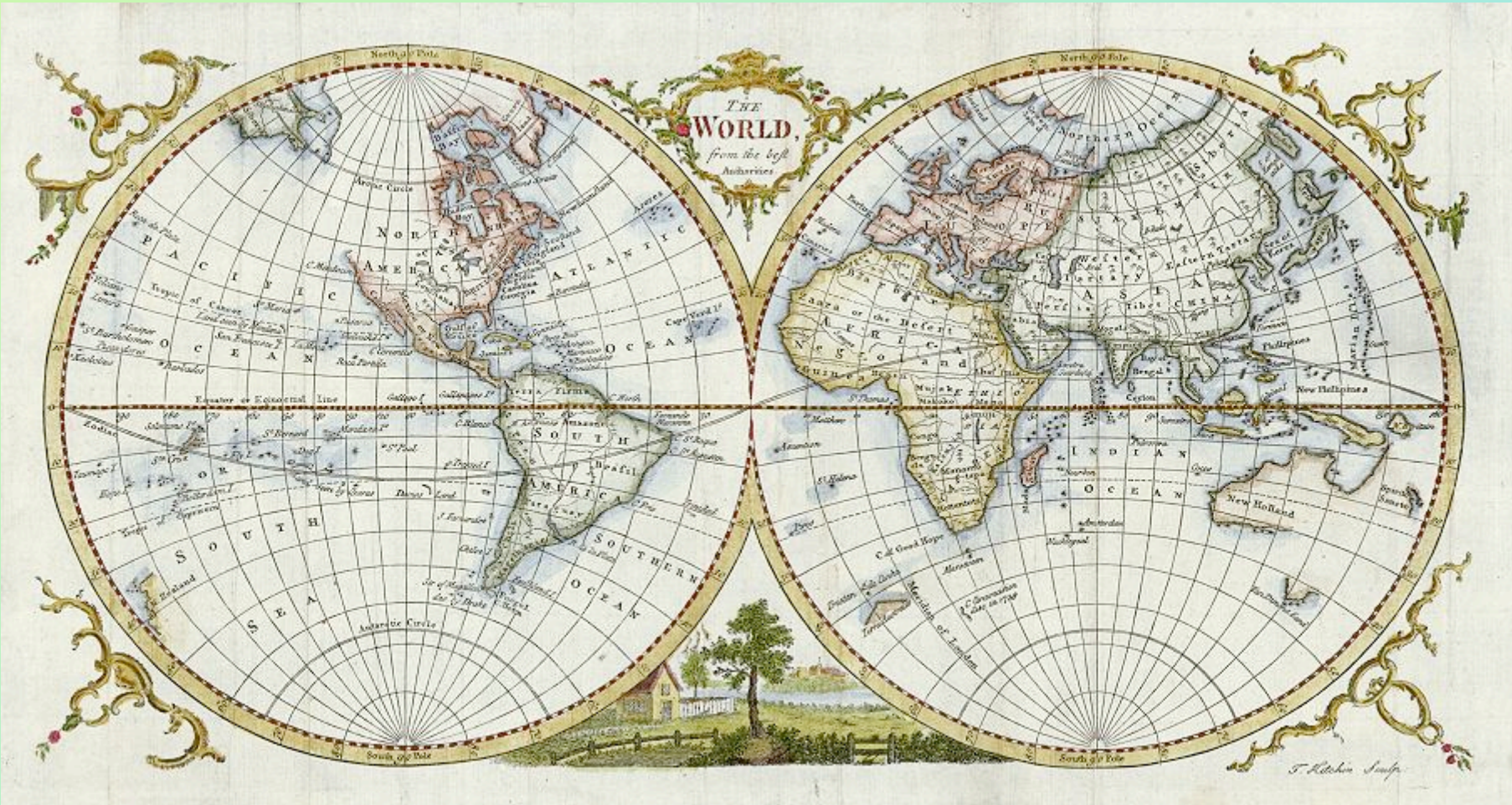

Maps are Useful!

Constellations

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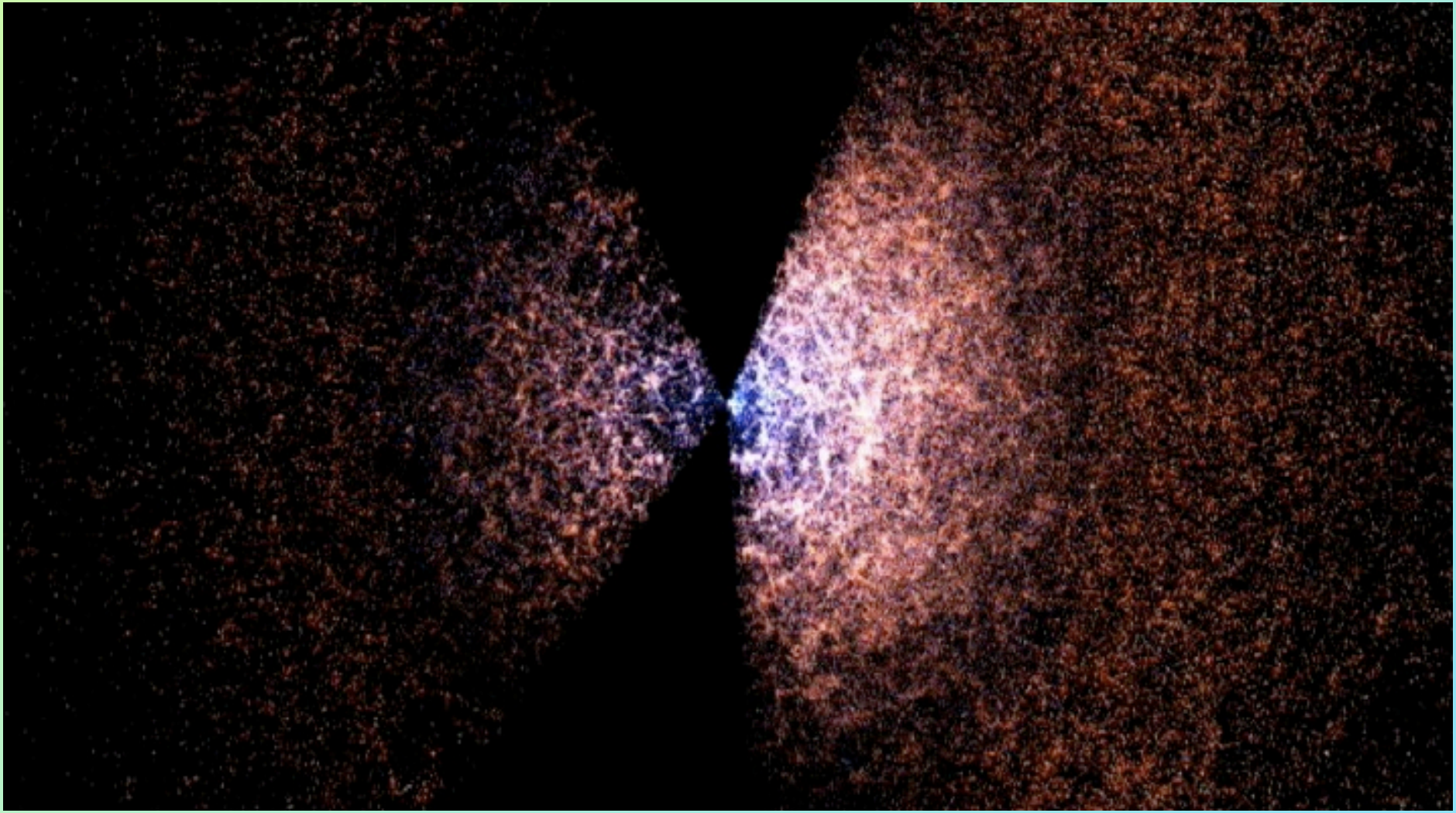

SDSS

Science or Stamp Collecting?

Context: Proposal for SDSS $\Rightarrow$ SDSS-II

- **Dominant view on how to obtain funding:**
  - » **Parametric Cosmology**
    - – **dark energy, m_ν , …**
- **Realistic view of how science works:**
  - » **Mapping the Sky / Astronomical Infrastructure**
  - » **Quality maps will produce large scientific yield.**

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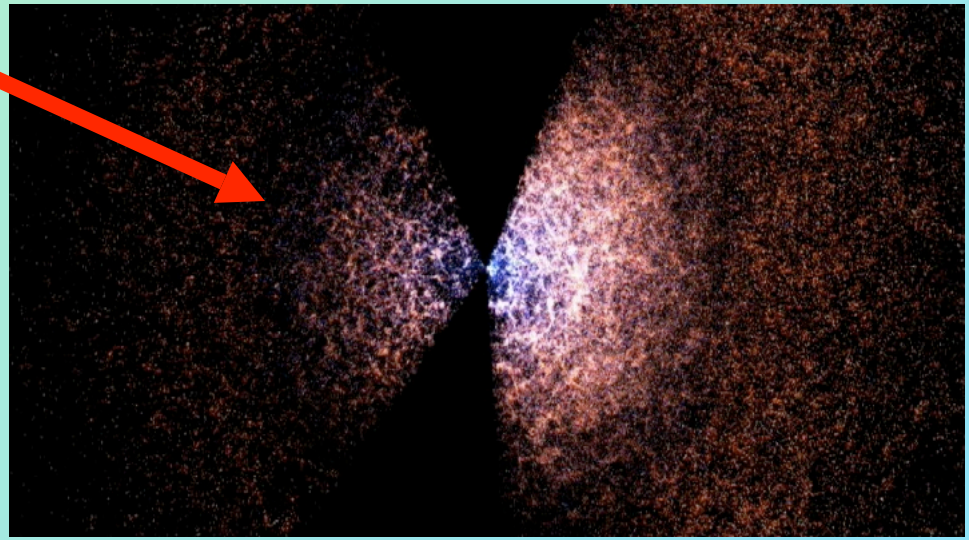
Make it and they will come!

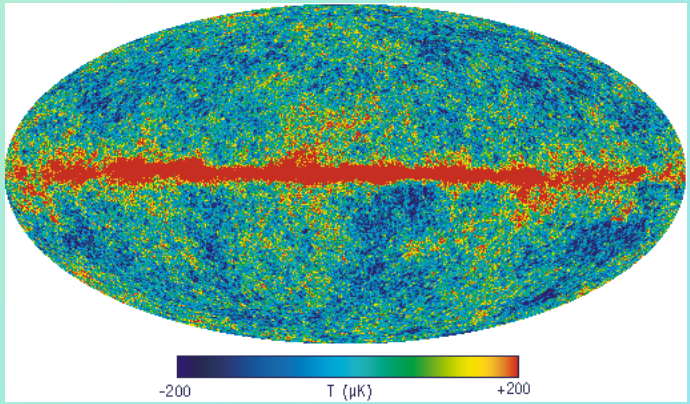
Quantify New Information Obtained by Better Mapping



Cartography and Parameter Estimation are Mathematically Similar

Measurements with Statistical Uncertainties!

- **Parametric Cosmology:**
  - » H_0, $T_{\gamma 0}$, Ω_{m0}, Ω_{b0}, m_ν, $\sigma 8$, n_s, n_t, r, w, w_A…
  - » ~10^1 pixels
- **Power Spectrum Estimation:**
  - » e.g. C^{TT}_ℓ
  - » ~10^3 pixels
- **Cartography:**
  - » ~10^6 pixels



Important Tool: Fisher Metric

a.k.a. Fisher Information Matrix

$$\begin{aligned} \boldsymbol{F}[\boldsymbol{p}] &= \langle (\boldsymbol{\nabla}_{\boldsymbol{x}} \ln[\mathcal{L}[\boldsymbol{x} \mid \boldsymbol{d}]])\, (\boldsymbol{\nabla}_{\boldsymbol{x}} \ln[\mathcal{L}[\boldsymbol{x} \mid \boldsymbol{d}]]) \rangle_{\boldsymbol{d}} \\ &= -\langle \boldsymbol{\nabla}_{\boldsymbol{x}}\, \boldsymbol{\nabla}_{\boldsymbol{x}} \ln[\mathcal{L}[\boldsymbol{x} \mid \boldsymbol{d}]] \rangle_{\boldsymbol{d}} \end{aligned}$$

$\boldsymbol{d}$ - data
$\boldsymbol{x}$ - parameters or "map"
$\mathcal{L}$ - likelihood function

- **Inverse Fisher Matrix gives lower bound on errors of estimators**

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- Inverse Fisher Matrix gives lower bound on errors of estimators
- Fisher Matrix transforms like a tensor on parameter space manifold!
  - » It is parameterization independent
- It is a positive-definite rank-2 tensor.
- Can be used as a metric on parameter space
  - » e.g. space of possible maps

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Fisher Geometry!

What is Fisher Geometry?

- **Fisher Distance:**

$$d_{\mathrm{F}}[\boldsymbol{x}_1, \boldsymbol{x}_2] = \int_{\mathrm{geodesic}} d\ell \, \sqrt{\boldsymbol{x}'[\ell] \cdot \boldsymbol{F}[\boldsymbol{x}[\ell]] \cdot \boldsymbol{x}'[\ell]}$$

- **For your measurement (technique) $d_{\mathrm{F}}[x_1, x_2]$ gives the "# of**

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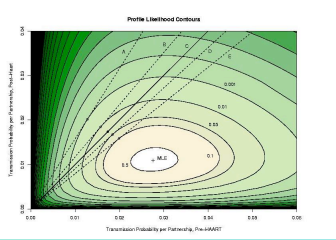
- **Fisher Geometry is "Physics Free"**
  - » **depends on limitation of measurement**
    - – **Noise (instrumental, sky, …)**
    - – **Sky coverage**
    - – **Overall limitations of technique.**

Fisher Volume

- **Fisher Volume for measurement (B)**

$$\mathcal{V}_{\mathrm{F}}^{(B)} = \int d^n \boldsymbol{x} \sqrt{\mathrm{Det}[\boldsymbol{F}_B[\boldsymbol{x}]]} \; \frac{\mathcal{L}_B[\boldsymbol{x}]}{\sup[\mathcal{L}_B]}$$

  - » **uninteresting**



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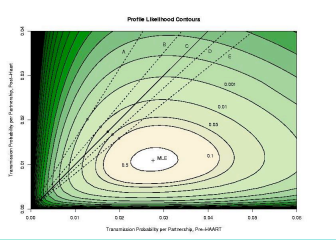

- **Conditional Fisher Volume (measurement A given B)**

$$\mathcal{V}_{\mathrm{F}}^{(A|B)} = \int d^n \boldsymbol{x} \sqrt{\mathrm{Det}[\boldsymbol{F}_{A+B}[\boldsymbol{x}]]} \; \frac{\mathcal{L}_B[\boldsymbol{x}]}{\sup[\mathcal{L}_B[\boldsymbol{x}]]}$$

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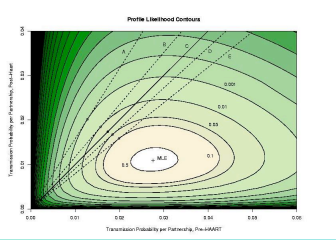

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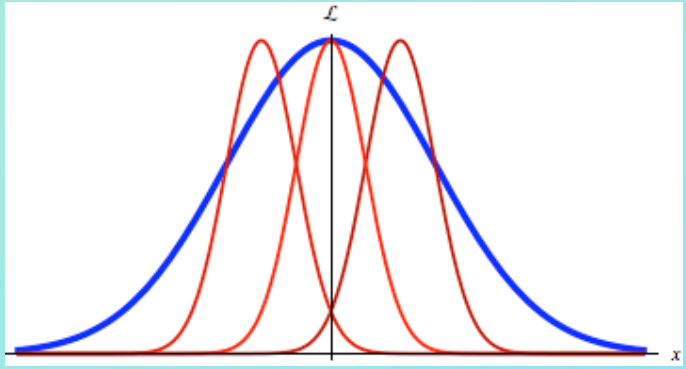
Roughly: # of distinguishable results obtainable from A+B which were consistent with measurement B.

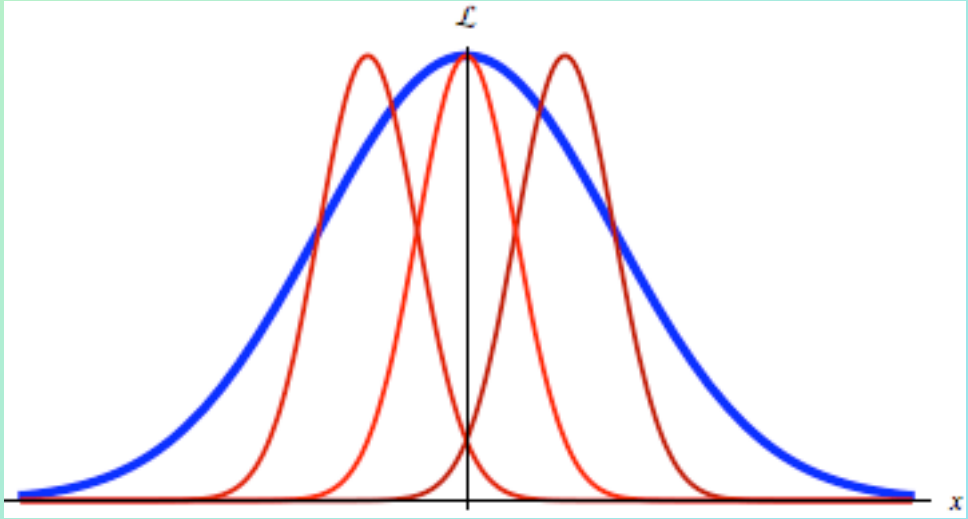


Fisher Information Number

clean it up

Natural log of # of distinguishable results obtainable from A+B which are consistent with measurement B.

$$I_{\mathrm{F}}^{(A|B)} = \ln\left[\frac{\mathcal{V}_{\mathrm{F}}^{(A|B)}}{\mathcal{V}_{\mathrm{F}}^{(B)}}\right]$$



Fisher Information Number

Examples

- Unconstrained map w/ Gaussian measurement "noise".
  - » 2 measurements of same map with noise matrices $\boldsymbol{N}_A$ and $\boldsymbol{N}_B$.

$$I_{\text{unconstrained}}^{(A|B)} = \frac{1}{2}\ln\left[\mathrm{Det}\left[\boldsymbol{I} + \boldsymbol{N}_B \cdot \boldsymbol{N}_A{}^{-1}\right]\right]$$

Fisher Information Number

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- Gaussian map w/ Gaussian "noise" w/ known Signal
  - » Signal matrix $\boldsymbol{S}$ (e.g. $P[k]$ or $\boldsymbol{C}_l$) with noise matrices $\boldsymbol{N}_A$, $\boldsymbol{N}_B$.

$$I^{(A|B)}_{\text{Gaussian}} = \frac{1}{2}\ln\left[\frac{\mathrm{Det}\left[\boldsymbol{I}+\boldsymbol{S}\cdot\boldsymbol{N}_A{}^{-1} + \boldsymbol{S}\cdot\boldsymbol{N}_B{}^{-1}\right]}{\mathrm{Det}\left[\boldsymbol{I} + \boldsymbol{S}\cdot\boldsymbol{N}_B{}^{-1}\right]}\right]$$

Fisher Information Number

Examples

- Unconstrained map w/ Gaussian measurement "noise".
  - » 2 measurements of same map with noise matrices N_A and N_B.

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- Gaussian map w/ Gaussian "noise" w/ known Signal
  - » Signal matrix S (e.g. $P[k]$ or C_l) with noise matrices N_A, N_B.

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- N.B.

$$\lim_{\frac{S}{N}\to\infty} I^{(A|B)}_{\text{Gaussian}} = I^{(A|B)}_{\text{unconstrained}}$$

Fisher Information Number

Warning

- Fisher geometry is Physics Free (unbiased)
  - » Adding a new significant digit to the value of a well measured pixel is contributes just as much as measuring the 1st significant digit of a "new" pixel
    - A scientist might find one more useful than the other
  - » It counts all pixels and angular scales equally.
    - A scientist may find some things more interesting than others.
- Fisher information is geometric (unbiased)
  - » It doesn't matter how you parameterize your map you will always get the same number.
    - If 2 quantities have a 1-1 relation, (e.g. density - potential) you get the same number.
      - This works even if the relation is non-linear!

Fisher Information Number

Examples

- Gaussian all sky maps

$$I^{(A|B)}_{\text{Gaussian}} = \frac{1}{2}\sum_l (2\,l+1)\ln\left[\frac{1+\frac{C_l^S}{C_l^{N,A}}+\frac{C_l^S}{C_l^{N,B}}}{1+\frac{C_l^S}{C_l^{N,B}}}\right]$$

$$I^{(A|B)} \sim \frac{\left(\frac{\sigma^B_{\text{beam}}}{\sigma^A_{\text{beam}}}\right)^2 - 1}{{\sigma^A_{\text{beam}}}^2} \sim 10^{5-6}$$

$$e^{I^{(A|B)}} \sim 10^{10^5}$$ *possibilities*

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- For hi-S/N, beam limited (not noise limited) surveys

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$$e^{I^{(A|B)}} \sim 10^{10^5}$$ *possibilities*

» Increased number of resolved hi-S/N pixels dominates

Fisher Information Number

Summary

- Using the Fisher information metric a "geometrical" measure of the amount of information obtained by making new measurements.
  - » The information measure independent of how one parameterized what one is measuring.
- This information measure is "physics free".
  - » This can be good - no scientist bias.
  - » This can be bad - justify stamp-collecting.
- One can compute it for almost any type of measurement which one has a statistical model for
  - » It may be difficult to compute!
  - » It is easy for Gaussian systems (much of cosmology).